The analysis of the channel-flow DNS data performed by Mansour, Kim, and Moin (1988) suggests strongly that, for such near-wall flows, it is preferable to use the decomposition in terms of $\mathcal{R}_{ij}^{(a)}$ rather than that in terms of $\mathcal{R}_{ij}$. It is found that the pressure-transport of kinetic energy $\frac{1}{2}\Pi_{ii}$ is quite small (see Figs. 7.18 and 7.34 on pages 294 and 322), so that in fact Π_{ij} is itself almost redistributive. The profiles of Π_{ij} (Figs. 7.35–7.38 on pages 324–325) exhibit simple behavior with Π_{ij} being zero at the wall (because $\mathbf{u}$ is zero there). In contrast, $\mathcal{R}_{22}$ and $\partial T_{222}^{(p)}/\partial y$ exhibit much more complicated behavior, including several sign changes. Although both terms are zero at the wall, within the viscous sublayer they are large, but almost cancel. In view of these considerations, we henceforth use the decomposition Eq. (11.5) in terms of $\mathcal{R}_{ij}^{(a)}$ and $\mathbf{T}^{(p)}$.

For inhomogeneous flows, the redistribution $\mathcal{R}_{ij}^{(a)}$ is modelled in terms of local quantities. That is, $\mathcal{R}_{ij}^{(a)}(\mathbf{x}, t)$ is modelled in terms of $\langle u_i u_j \rangle$, $\partial\langle U_i \rangle/\partial x_j$ and ε, evaluated at $(\mathbf{x}, t)$, just as in homogeneous turbulence (Eq. 11.135). In terms of the ease of solution of the resulting model equations, this is certainly an expedient assumption compared to the alternative of including non-local quantities. But it should be recognized that $p'(\mathbf{x}, t)$ is governed by a Poisson equation (Eq. 11.9), so that it is influenced by quantities such as $\partial\langle U_i \rangle/\partial x_j$ some distance from $\mathbf{x}$. As a consequence, the modelling of $\mathcal{R}_{ij}^{(a)}$ in inhomogeneous flows is less secure than it is for homogeneous turbulence.

(In contrast, in elliptic relaxation models—Section 11.8—the modelling of $\mathcal{R}_{ij}^{(a)}$ is non-local.)

11.6.2 Reynolds-Stress Transport

With the velocity-pressure-gradient tensor Π_{ij} decomposed according to Eq. (11.5), the exact evolution equation for the Reynolds stresses is

$$\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t}\langle u_i u_j \rangle + \frac{\partial}{\partial x_k}\left[T_{kij}^{(\nu)} + T_{kij}^{(p')} + T_{kij}^{(u)}\right] = \mathcal{P}_{ij} + \mathcal{R}_{ij}^{(a)} - \varepsilon_{ij}, \tag{11.138}$$

where the three fluxes are: viscous diffusion

$$T_{kij}^{(\nu)} = -\nu \frac{\partial\langle u_i u_j \rangle}{\partial x_k}, \tag{11.139}$$

pressure transport

$$T_{kij}^{(p')} = \tfrac{2}{3}\delta_{ij}\langle u_k p' \rangle/\rho, \tag{11.140}$$

and turbulent convection

$$T_{kij}^{(u)} = \langle u_i u_j u_k \rangle. \tag{11.141}$$

Viscous diffusion is negligible except in the viscous wall region; and, since the term is in closed form, it requires no further discussion.

Pressure Transport. Before the advent of DNS, there was little reliable information about pressure correlations. Based on an analysis of nearly-homogeneous turbulence, Lumley (1978) proposed the model

$$\frac{1}{\rho}\langle u_i p' \rangle = -\tfrac{1}{5}\langle u_i u_j u_j \rangle. \tag{11.142}$$

Since the pressure-transport given by Eq. (11.140) is isotropic, it can be examined through the corresponding term in the kinetic energy equation

$$\tfrac{1}{2}T_{kii}^{(p')} = \langle u_k p' \rangle / \rho, \tag{11.143}$$

for which Lumley's model is

$$\tfrac{1}{2}T_{kii}^{(p')} = -\tfrac{1}{5}\langle u_k u_i u_i \rangle. \tag{11.144}$$

This is $-\frac{2}{5}$ of the convective flux

$$\tfrac{1}{2}T_{kii}^{(u)} = \tfrac{1}{2}\langle u_k u_i u_i \rangle. \tag{11.145}$$

The kinetic energy budgets for channel flow (Fig. 7.18 on page 294) and for the boundary layer (Fig. 7.34 on page 322) show that pressure transport is not very significant close to the wall, and that Lumley's model is qualitatively incorrect. At the edge of the boundary layer, however, the pressure transport is more important, and Lumley's model is at least qualitatively correct.

To examine the pressure transport for free shear flows, the kinetic energy budget for the self-similar temporal mixing layer is shown in Fig. 11.16. It may be seen from Fig. 11.16(a) that the pressure transport is relatively small over most of the layer, and that Lumley's model is quite reasonable.

The edge of the layer is examined in more detail in Fig. 11.16(b). The rotational turbulent fluctuations within the layer induce irrotational fluctuations in the non-turbulent region ($y/\delta > 1$, say). This transfer of energy is effected by the fluctuating pressure field, and hence it appears in the kinetic energy budget as pressure transport. As may be seen from the figure, at the edge of the layer this pressure transport becomes dominant.

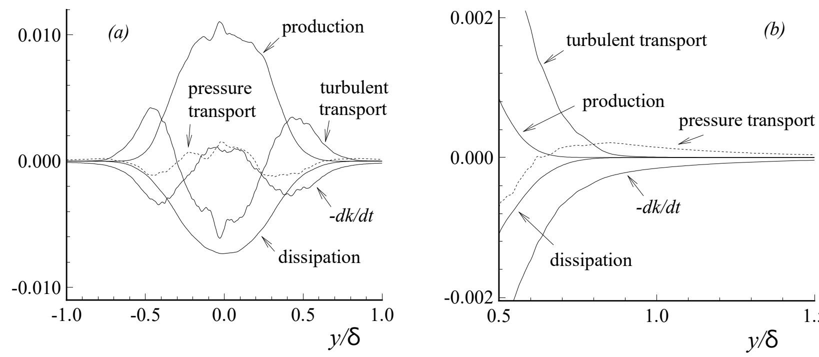

Figure 11.16: Kinetic energy budget in the temporal mixing layer from the DNS data of Rogers and Moser (1994): (a) across the whole flow (b) an expanded view of the edge of the layer. The contributions to the budget are: production $\mathcal{P}$; dissipation $-\varepsilon$; rate of change $-\mathrm{d}k/\mathrm{d}t$; turbulent transport; pressure transport (dashed line). All quantities are normalized by the velocity difference and the layer thickness δ (see Fig. 5.21 on page 145).

Demuren et al. (1996) use DNS data to examine separately the pressure transport due to the slow and rapid pressure; and models for each contribution are proposed.

In most Reynolds-stress models the pressure transport is either neglected or (implicitly or explicitly) it is modelled together with the turbulent convection by a gradient-diffusion assumption.

Gradient-Diffusion Models. The simplest gradient-diffusion model for $T'_{kij} = T^{(u)}_{kij} + T^{(p')}_{kij}$, due to Shir (1973), is

$$T'_{kij} = -C_s \frac{k^2}{\varepsilon} \frac{\partial \langle u_i u_j \rangle}{\partial x_k}, \tag{11.146}$$

where C_s is a model constant. In more general use is the model of Daly and Harlow (1970) which uses the Reynolds-stress tensor to define an anisotropic diffusion coefficient:

$$T'_{kij} = -C_s \frac{k}{\varepsilon} \langle u_k u_\ell \rangle \frac{\partial \langle u_i u_j \rangle}{\partial x_\ell}. \tag{11.147}$$

For this model Launder (1990) suggests the value of the constant $C_s = 0.22$.

If $T^{(u)}_{kij} \equiv \langle u_k u_i u_j \rangle$ is to be modelled separately, then a consistent model is required to be symmetric with respect to all three indices. Such symmetric models, necessarily involving cross-diffusion, have been proposed by Mellor and Herring (1973)

$$\langle u_i u_j u_k \rangle = -C_s \frac{k^2}{\varepsilon} \left(\frac{\partial \langle u_j u_k \rangle}{\partial x_i} + \frac{\partial \langle u_i u_k \rangle}{\partial x_j} + \frac{\partial \langle u_i u_j \rangle}{\partial x_k} \right), \tag{11.148}$$

and by Hanjalić and Launder (1972)

$$\langle u_i u_j u_k \rangle = -C_s \frac{k}{\varepsilon} \times \left(\langle u_i u_\ell \rangle \frac{\partial \langle u_j u_k \rangle}{\partial x_\ell} + \langle u_j u_\ell \rangle \frac{\partial \langle u_i u_k \rangle}{\partial x_\ell} + \langle u_k u_\ell \rangle \frac{\partial \langle u_i u_j \rangle}{\partial x_\ell} \right). \tag{11.149}$$

However, consistent models for T'_{kij} are required to be symmetric with respect to i and j only—a requirement that is satisfied by all four of the above models.

An examination of the transport equation for the triple correlation $\langle u_i u_j u_k \rangle$ can motivate yet more elaborate models, involving the mean velocity gradients. But the general experience of practitioners is that the modelling of T'_{kij} is not a critical ingredient in the overall model, and that the relatively simple Daly-Harlow model is adequate (Launder 1990). This view is questioned by Parneix, Laurence, and Durbin (1998) who suggest that deficiencies in the transport model are responsible for inaccuracies in the calculation of the flow over a backward facing step. Direct tests of the models for $T^{(u)}_{kij}$ against experimental data can be found in Schwarz and Bradshaw (1994) and references therein.

11.6.3 Dissipation Equation

The standard model equation for ε used in Reynolds-stress models is that proposed by Hanjalić and Launder (1972)

$$\frac{\bar{\mathrm{D}}\varepsilon}{\bar{\mathrm{D}}t} = \frac{\partial}{\partial x_i} \left(C_\varepsilon \frac{k}{\varepsilon} \langle u_i u_j \rangle \frac{\partial \varepsilon}{\partial x_j} \right) + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}, \tag{11.150}$$

with $C_\varepsilon = 0.15$, $C_{\varepsilon 1} = 1.44$ and $C_{\varepsilon 2} = 1.92$ (Launder 1990). There are two differences between this equation and that used in the k-ε model (Eq. 10.53). First, the production $\mathcal{P}$ is evaluated directly from the Reynolds stresses rather than as $2\nu_T \bar{S}_{ij} \bar{S}_{ij}$. Second, the diffusion term involves an anisotropic diffusivity.

As mentioned in Section 10.4.3, several modifications to the dissipation equation have been proposed. In a Reynolds-stress model, the invariants of